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A STOCHASTIC REFORMULATION OF THE POWER FLOW EQUATIONS FOR MEMBRANES AND PLATES

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1. Introduction

A reformulation is presented of the power flow equations obtained by Bouthier and Bernhard [1, 2] for membranes and plates. An explicit stochastic formulation is carried out in order to write, in the high frequency domain, a simple relationship between the expectations of the time averaged power flow and energy density. It will be shown that this relationship is the same as the one obtained by Bouther and Bernhard [1, 2] between the "smoothed" time average energy density and power flow. The notion of *smoothing operation* developed by Bouthier and Bernhard will be explicitly defined in the following.

Besides reformulation of the energy equations, the problem of the evaluation of the power input is considered. It will be shown that the exact power input (which is generally not reachable) may be replaced by the power input of the infinite associated system.

2. Space averaged energetics of plates and membranes

In two recent articles, Bouthier and Bernhard [1, 2] proposed an energy flow formulation to treat the high frequency dynamic behaviour of membranes and plates. The basis of the theory is the steady state energy balance, similar for both plates and membranes:

$$\pi_{inj} = \pi_{diss} + \vec{\nabla} \cdot \vec{I}. \tag{1}$$

π_{inj} and π_{diss} denote respectively the input power density and the dissipated power density. $\vec{I}$ is the instantaneous intensity (W/m^2), and is expressed in terms of the displacement $\vec{u}$ and the stress tensor $\bar{\bar{\sigma}}$ as

$$\vec{I} = -\bar{\bar{\sigma}} \colon \partial\vec{u}/\partial t \tag{2}$$

The aim of Bouthier and Bernhard was to express the different variables of equation (1), using the energy density variable, sum of the kinetic and potential energy densities (J/m^2). The solutions of the governing equations of the membrane and the plates were written in terms of a plane wave approximation. Moreover, the evanescent wave field present in the plate was neglected. Finally the displacement solutions was expressed as

$$U(x, y, t) = (A_x\,\mathrm{e}^{-\mathrm{i}k_x x} + B_x\,\mathrm{e}^{\mathrm{i}k_x x})(A_y\,\mathrm{e}^{-\mathrm{i}k_y y} + B_y\,\mathrm{e}^{\mathrm{i}k_y y})\,\mathrm{e}^{\mathrm{i}\omega t}. \tag{3}$$

k_x and k_y represent the wave number components. One can then write the expressions of the time averaged energy density and intensity, in terms of the displacement and its 1derivatives. A smoothing operation is then performed on the time averaged intensity and energy density. This operation may be understood as a space average, even if some

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authors [3] have doubts about the interpretation that can be given to this calculus. This operation is defined as

$$\underline{q} = \frac{4}{\lambda_x \lambda_y} \int_{x - \lambda_x/4}^{x + \lambda_x/4} \int_{y - \lambda_y/4}^{y + \lambda_y/4} q \, \mathrm{d}x \, \mathrm{d}y, \tag{4}$$

where q is one of the energy variables, and λ_x and λ_y represent the components of the wavelength λ. The smoothed variables are underlined, while the time averaged variables are written within brackets. A relationship between the smoothed expressions of the time averaged energy density and intensity $\langle \underline{e} \rangle$ and $\langle \underline{\vec{I}} \rangle$ is obtained [1, 2]:

$$\langle \underline{\vec{I}} \rangle = -\frac{c_g^2}{\eta\omega}\left(\frac{\partial \langle \underline{e} \rangle}{\partial x}\vec{i} + \frac{\partial \langle \underline{e} \rangle}{\partial y}\vec{j}\right). \tag{5}$$

c_g is the group velocity, η is the hysteretic damping factor and ω denotes the circular frequency. One can now explicit the energy dissipation variable in terms of the energy density [2, 4], using the approximate relationship

$$\langle \underline{\pi}_{diss} \rangle \approx \eta\omega \langle \underline{e} \rangle. \tag{6}$$

Finally, a differential equation for the smoothed time averaged energy density is obtained. For the membrane as well as for the plate, the equation may be written as

$$(-c_g^2/\eta\omega)\Delta\langle \underline{e} \rangle + \eta\omega\langle \underline{e} \rangle = \langle \pi_{inj} \rangle. \tag{7}$$

The energy formulation summarized in this section is an important improvement on the Statistical Energy Analysis (SEA) developed by Lyon [5] in the early sixties. One of the main advantages of the energy flow formulation is its ability to provide local information about the energy levels, unlike the SEA which may only give a global result for each sub-system of a structure. However, some questions remain concerning the power flow equations. The first difficulty is the evaluation of the influence of the smoothing operation. Bouthier and Bernhard noticed that this operation seemed not to be valid when the frequency was not sufficiently high.

In what follows, the authors introduce a stochastic parameter of the geometrical characteristics of the plates and membranes, and will prove that one can write a relationship similar to the one obtained by Bouthier and Bernhard, valid in the high frequency domain, between the random mean energy density and the random mean intensity. The geometrical random variables illustrate the high sensitivity of the dynamic structural responses to small perturbations of the mechanical and geometrical parameters, in the high frequency domain.

Generally, energy methods meet another important difficulty concerning the evaluation of the power input. Bouthier and Bernhard avoided this difficulty by considering a pure energy source without relating it to the associated force source. Unfortunately, experimental datas and simulations propose generally force–displacement inputs and one must convert them to an energy input. The authors will show that the random mean power input is equivalent to the power input of the infinite associated system, in the high frequency domain.

3. STOCHASTIC MEMBRANE POWER FLOW EQUATIONS

The governing equation of a membrane under an in-plane tensile force T and submitted to a punctual harmonic loading F, elastic linear properties and small harmonic deflections being assumed, is expressed as [6]

$$T\Delta U + \rho\omega^2 U = -F\,\delta(x_0, y_0), \tag{8}$$

where ρ is the mass per unit area and U is the deflection of the membrane. A loss factor η is introduced in equation (8). The governing equation may be written as

$$T(1+\mathrm{i}\eta)\Delta U + \rho\omega^2 U = -F\,\delta(x_0, y_0). \tag{9}$$

The solution of equation (9) is approximated to a plane wave field and is written as

$$U(x,y) = (A_x\,\mathrm{e}^{-\mathrm{i}k_x x} + B_x\,\mathrm{e}^{\mathrm{i}k_x x})(A_y\,\mathrm{e}^{-\mathrm{i}k_y y} + B_y\,\mathrm{e}^{\mathrm{i}k_y y}); \tag{10}$$

k_x and k_y are the wave number components of k. The following notations are introduced:

$$|k|^2 = \omega^2\rho/T \tag{11}$$

and

$$k_x = k_{x1}(1-\mathrm{i}\eta/2), \qquad k_y = k_{y1}(1-\mathrm{i}\eta/2), \qquad c_g = \sqrt{T/\rho}. \tag{12, 13}$$

c_g represents the wave group velocity of the membrane. The time averaged energy density and the components of the intensity are

$$\langle \mathrm{e}\rangle_t = \frac{T}{4}\left\{\frac{\partial U}{\partial x}\left(\frac{\partial U}{\partial x}\right)^* + \frac{\partial U}{\partial y}\left(\frac{\partial U}{\partial y}\right)^* + \frac{1}{c_g^2}\frac{\partial U}{\partial t}\left(\frac{\partial U}{\partial t}\right)^*\right\}, \tag{14}$$

$$\langle I_x\rangle_t = \frac{T}{2}\left\{\frac{\partial U}{\partial x}\left(\frac{\partial U}{\partial t}\right)^*\right\}, \qquad \langle I_y\rangle_t = \frac{T}{2}\left\{\frac{\partial U}{\partial y}\left(\frac{\partial U}{\partial t}\right)^*\right\}. \tag{15}$$

The time average is represented by the symbol $\langle\ \rangle_t$. A stochastic parameter is now introduced into the co-ordinates of the point of measurement. The random co-ordinates $\tilde{x}$ and $\tilde{y}$ are written as

$$\tilde{x} = x + \epsilon_x, \qquad \tilde{y} = y + \epsilon_y. \tag{16}$$

ϵ_x and ϵ_y are two independent zero mean random Gaussian variables, and x and y are the two deterministic co-ordinates. A_x, B_x, A_y and B_y do not depend on the value of ϵ_x and ϵ_y; they are functions only of the position of the boundaries. The explicit expressions for the time averaged energy density and intensity components are given partially. The entire calculus can be found in the Ph.D. Thesis of Bouthier [7]. The time averaged energy density is

$$\begin{aligned}\langle\tilde{e}\rangle_t = {} & (T/2)(|k_x|^2+|k_y|^2)\{|A_x|^2|A_y|^2\,\mathrm{e}^{-\eta(k_{x1}\tilde{x}+k_{y1}\tilde{y})} + |A_x|^2|B_y|^2\,\mathrm{e}^{-\eta(k_{x1}\tilde{x}-k_{y1}\tilde{y})} \\ & + |B_x|^2|A_y|^2\,\mathrm{e}^{\eta(k_{x1}\tilde{x}-k_{y1}\tilde{y})} + |B_x|^2|B_y|^2\,\mathrm{e}^{\eta(k_{x1}\tilde{x}+k_{y1}\tilde{y})}\} \\ & + (T_4)|k_x|^2|A_x|^2 B_y A_g^*\,\mathrm{e}^{-\eta k_{x1}\tilde{x}}\,\mathrm{e}^{2\mathrm{i}k_{y1}\tilde{y}} - (T/4)|k_x|^2|A_y|^2 B_x A_x^*\,\mathrm{e}^{-\eta k_{y1}\tilde{y}}\,\mathrm{e}^{-2\mathrm{i}k_{x1}\tilde{x}} \\ & - (T/4)|k_x|^2 B_x B_y A_x^* A_y^*\,\mathrm{e}^{2\mathrm{i}(k_{x1}\tilde{x}+k_{y1}\tilde{y})} + \cdots,\end{aligned} \tag{17}$$

The time averaged intensity components are

$$\begin{aligned}\langle\tilde{I}_x\rangle_t = {} & (T/2)k_x\omega\{|A_x|^2|A_y|^2\,\mathrm{e}^{-\eta(k_{x1}\tilde{x}+k_{y1}\tilde{y})} + |A_x|^2|B_y|^2\,\mathrm{e}^{-\eta(k_{x1}\tilde{x}-k_{y1}\tilde{y})} \\ & - |B_x|^2|A_y|^2\,\mathrm{e}^{\eta(k_{x1}\tilde{x}-k_{y1}\tilde{y})} - |B_x|^2|B_y|^2\,\mathrm{e}^{\eta(k_{x1}\tilde{x}+k_{y1}\tilde{y})}\} \\ & + (T/2)\,\mathrm{Re}\,\{k_x\omega|A_x|^2 A_y B_y^*\,\mathrm{e}^{-\eta k_{x1}\tilde{x}}\,\mathrm{e}^{-2\mathrm{i}k_{y1}\tilde{y}} \\ & + k_x\omega A_x|A_y|^2 B_x^*\,\mathrm{e}^{-\eta k_{y1}\tilde{y}}\,\mathrm{e}^{-2\mathrm{i}k_{x1}\tilde{x}} + \cdots\},\end{aligned} \tag{18}$$

and

$$\langle \tilde{I}_y \rangle_t = (T/2)k_y\omega\{|A_y|^2|A_y|^2\,\mathrm{e}^{-\eta(k_{x1}\tilde{x}+k_{y1}\tilde{y})} + |A_x|^2|B_y|^2\,\mathrm{e}^{-\eta(k_{x1}\tilde{x}-k_{y1}\tilde{y})} - |B_x|^2|A_y|^2\,\mathrm{e}^{\eta(k_{x1}\tilde{x}-k_{y1}\tilde{y})} - |B_x|^2|B_y|^2\,\mathrm{e}^{\eta(k_{x1}\tilde{x}+k_{y1}\tilde{y})}\} + (T/2)\,\mathrm{Re}\,\{k_y\omega|A_x|^2A_yB_y^*\,\mathrm{e}^{-\eta k_{x1}\tilde{x}}\,\mathrm{e}^{-2ik_{y1}\tilde{y}} + k_y\omega A_x|A_y|^2B_x^*\,\mathrm{e}^{-\eta k_{y1}\tilde{y}}\,\mathrm{e}^{-2ik_{x1}\tilde{x}} + \cdots\}, \quad (19)$$

It is now possible to evaluate the expectations of the energy variables expressed previously. The averaging operation is explicitly carried out for two different terms. The developments for the other terms are not given, as the calculus is too lengthy and repetitive. The average of one of the terms may be calculated:

$$\langle |A_y|^2B_xA_x^*\,\mathrm{e}^{-\eta k_{y1}\tilde{y}}\,\mathrm{e}^{-2ik_{x1}\tilde{x}}\rangle_{xy} = |A_y|^2B_xA_x^*\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{e}^{-\eta k_{y1}(y+\epsilon_y)}\,\mathrm{e}^{-2ik_{x1}(x+\epsilon_x)} \times \frac{\mathrm{e}^{-(\epsilon_x^2+\epsilon_y^2)/2\sigma^2}}{\sigma^2 2\pi}\,\mathrm{d}\epsilon_x\,\mathrm{d}\epsilon_y. \quad (20)$$

The expectation operation over x and y is represented by the symbol $\langle\ \rangle_{xy}$. σ represents the standard deviation of the Gaussian law. The term $\eta\epsilon_y$ as well as $\eta\epsilon_x$ are assumed to be second order terms compared to ηx and ηy and are not taken into account. Finally, equation (20) gives

$$\langle |A_y|^2B_xA_x^*\,\mathrm{e}^{-\eta_{y1}\tilde{y}}\,\mathrm{e}^{-2ik_{x1}\tilde{x}}\rangle_{xy} = |A_y|^2B_xA_x^*\,\mathrm{e}^{-\eta k_{y1}y}\,\mathrm{e}^{-2ik_{x1}x}\,\mathrm{e}^{-2k_{x1}^2\sigma^2}. \quad (21)$$

The average of another term is calculated by using the same procedure:

$$\langle B_xB_yA_x^*A_y^*\,\mathrm{e}^{2i(k_{x1}\tilde{x}+k_{y1}\tilde{y})}\rangle_{xy} = B_xB_yA_x^*A_y^*\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{e}^{2i(k_{x1}(x+\epsilon_x)+k_{y1}(y+\epsilon y))} \times \frac{\mathrm{e}^{-(\epsilon_x^2+\epsilon_y^2)/2\sigma^2}}{\sigma\sqrt{2\pi}}\,\mathrm{d}\epsilon_x\,\mathrm{d}\epsilon_y = B_xB_yA_x^*A_y^*\,\mathrm{e}^{2i(k_{x1}x+k_{y1}y)}\,\mathrm{e}^{-2(k_{x1}^2+k_{y1}^2)\sigma^2}. \quad (22)$$

The averaged values of the terms containing imaginary components may be neglected in the high frequency range. The terms multiplied by $\exp[-k_{x1}^2\sigma^2]$ or (and) $\exp[-k_{y1}^2\sigma^2]$ vanish in the high frequency field. The expectations of the time averages of the energy density and intensity components in the high frequency range may be approximated by

$$\langle\langle \tilde{e}\rangle_t\rangle_{xy} \approx (T/2)(|k_x|^2+|k_y|^2)\{|A_x|^2|A_y|^2\,\mathrm{e}^{-\eta(k_{x1}\tilde{x}+k_{y1}\tilde{y})} + |A_x|^2|B_y|^2\,\mathrm{e}^{-\eta(k_{x1}\tilde{x}-k_{y1}\tilde{y})} + |B_x|^2|A_y|^2\,\mathrm{e}^{\eta(k_{x1}\tilde{x}-k_{y1}\tilde{y})} + |B_x|^2|B_y|^2\,\mathrm{e}^{\eta(k_{x1}\tilde{x}-k_{y1}\tilde{y})}\}. \quad (23)$$

The expectations of the time averaged intensity components are

$$\langle\langle \tilde{I}_x\rangle_t\rangle_{xy} \approx (T/2)k_x\omega\{|A_x|^2|A_y|^2\,\mathrm{e}^{-\eta(k_{x1}\tilde{x}+k_{y1}\tilde{y})} + |A_x|^2|B_y|^2\,\mathrm{e}^{-\eta(k_{x1}\tilde{x}-k_{y1}\tilde{y})} - |B_x|^2|A_y|^2\,\mathrm{e}^{\eta(k_{x1}\tilde{x}-k_{y1}\tilde{y})} - |B_x|^2B_y|^2\,\mathrm{e}^{\eta(k_{x1}\tilde{x}+k_{y1}\tilde{y})}\} \quad (24)$$

and

$$\langle\langle \tilde{I}_y\rangle_t\rangle_{xy} \approx (T/2)k_y\omega\{|A_x|^2|A_y|^2\,\mathrm{e}^{-\eta(k_{x1}\tilde{x}+k_{y1}\tilde{y})} + |A_x|^2|B_y|^2\,\mathrm{e}^{-\eta(k_{x1}\tilde{x}-k_{y1}\tilde{y})} - |B_x|^2|A_y|^2\,\mathrm{e}^{\eta(k_{x1}\tilde{x}-k_{y1}\tilde{y})} - |B_x|^2|B_y|^2\,\mathrm{e}^{\eta(k_{x1}\tilde{x}+k_{y1}\tilde{y})}\}. \quad (25)$$

The symbol $\langle\langle\ \rangle_t\rangle_{xy}$ represents the two operations of time averaging and space expectation. The following relationship between the intensity and the energy density can then be deduced from equations (11, 13, 23–25):

$$\langle\langle\vec{\tilde{I}}\rangle_t\rangle_{xy} = -(c_g^2/\eta\omega)\nabla\langle\langle\tilde{e}\rangle_t\rangle_{xy}. \tag{26}$$

It is then possible to obtain the energy governing equation:

$$(-c_g^2/\eta\omega)\Delta\langle\langle\tilde{e}\rangle_t\rangle_{xy} + \eta\omega\langle\langle\tilde{e}\rangle_t\rangle_{xy} = \langle\pi_{inj}\rangle_t. \tag{27}$$

At this stage, the input power is only time averaged. Its value depends only on the input force, the mechanical parameters and the position of the boundaries, and is not a function of (x, y), the point of measurement.

The evaluation of the power input to the structure by using the value of the injected force is not easy to handle. Effectively, the expression for the exact power input contains all the modal information: that is to say the structural complexity. It will now be proved that for any kind of membrane shape, the introduction of a random parameter into the co-ordinates of the boundaries, gives a very simple expression for the averaged injected power. For this demonstration, the case of a circular clamped membrane of radius R is considered, loaded by a punctual loading F situated at the center of the structure. The displacement solution is

$$U(r) = \frac{-\mathrm{i}F}{4T}\,\frac{\mathrm{H}_0^{(1)}(kR)\mathrm{H}_0^{(2)}(kr) - \mathrm{H}_0^{(2)}(kR)\mathrm{H}_0^{(1)}(kr)}{\mathrm{H}_0^{(1)}(kR) + \mathrm{H}_0^{(2)}(kR)}, \tag{28}$$

and the resulting radial traction is:

$$T(r) = T\frac{\partial U(r)}{\partial r} = \frac{\mathrm{i}kF}{4}\,\frac{\mathrm{H}_0^{(1)}(kR)\mathrm{H}_1^{(2)}(kr) - \mathrm{H}_0^{(2)}(kR)\mathrm{H}_1^{(1)}(kr)}{\mathrm{H}_0^{(1)}(kR) + \mathrm{H}_0^{(2)}(kR)}, \tag{29}$$

where $\mathrm{H}_0^{(1)}$, $\mathrm{H}_0^{(2)}$, $\mathrm{H}_1^{(1)}$ and $\mathrm{H}_1^{(2)}$ represent the Hankel functions of zeros and first order. It is not possible to evaluate the power input at the loading point. Its value is locally infinite, due to the displacement singularity. Instead of considering the loading position, the input power is calculated on a circle whose center is the source location. The radius of the circle, r_0, is considered sufficiently large so that the Hankel functions evaluated at r_0 may be replaced by their far field asymptotic values [8]. A general rule postulates that the far field is reached at a distance equal to the half of the wave length. The time averaged power input in the membrane through the contour defined by r_0 is

$$\langle\pi_{inj}\rangle_t = -(2\pi r_0 T\omega/2)\ \mathrm{Re}\,\{\mathrm{i}(\partial U(r_0)/\partial r)U^*(r_0)\}. \tag{30}$$

The asymptotic expression for $\langle\pi_{inj}\rangle_t$ is

$$\langle\pi_{inj}\rangle_t \approx \frac{F^2\omega}{8T}\,\mathrm{e}^{-\eta k_0 r_0}\,\frac{1 + \mathrm{e}^{-2\mathrm{i}k_0(R-r_0)}\,\mathrm{e}^{-\eta k_0(R-r_0)}}{1 + \mathrm{e}^{-2\mathrm{i}k_0 R}\,\mathrm{e}^{-\eta k_0 R}}. \tag{31}$$

Using the symmetry of the structure and appropriate polar co-ordinates, one can write the wave number with only one radial component. The following notation is introduced: $k = k_0(1 - \mathrm{i}\eta/2)$. The term $\exp[-\eta k_0 r_0]$ is neglected. This simplification is accurate if r_0 is small. Taking r_0 equal to the wave length λ introduces an error of $\exp[-2\pi\eta]$. A Taylor series expansion of equation (31) valid in the high frequency range, is written:

$$\langle\pi_{inj}\rangle_t \approx (F^2\omega/8T)(1 + \mathrm{e}^{-2\mathrm{i}k_0(R-r_0)}\,\mathrm{e}^{-\eta k_0(R-r_0)} - \mathrm{e}^{2\mathrm{i}k_0(R-r_0)}\,\mathrm{e}^{-\eta k_0(R-r_0)} + \cdots). \tag{32}$$

The stochastic operations presented before are carried out, with R and r_0 replaced by the random variables $\tilde{R} = R + \epsilon_R$ and $\tilde{r}_0 = r_0 + \epsilon_{r_0}$. The expectation of the time averaged power input is calculated for Gaussian random variables. The final expression is

$$\langle\langle \tilde{\pi}_{inj} \rangle_t \rangle_{x_0 y_0} \approx (F^2\omega/8T)(1 + \mathrm{e}^{-2\mathrm{i}k_0(R-r_0)}\,\mathrm{e}^{-4k_0^2\sigma^2}\,\mathrm{e}^{-\eta k_0(R-r_0)} - \mathrm{e}^{2\mathrm{i}k_0(R-r_0)}\,\mathrm{e}^{-4k_0^2\sigma^2}\,\mathrm{e}^{-\eta k_0(R-r_0)} + \cdots). \tag{33}$$

The expectation over the boundary co-ordinates is defined by the symbol $\langle\ \rangle_{x_0 y_0}$. The oscillating term vanishes when the frequency increases to infinity. One can then replace the expression for π_{inj} by

$$\langle\langle \tilde{\pi}_{inj} \rangle_t \rangle_{x_0 y_0} \approx F^2\omega/8T. \tag{34}$$

Equation (34) is a very simple expression for the mean value of π_{inj}. Even if this last relation has been obtained for circular membranes, this result can be extended to any shapes of membranes, since the final result does not contain any geometrical information. However, this expression is not valid near the location of the loading. The final expression for the energy governing equation is

$$(-c_g^2/\eta\omega)\Delta\langle\langle\langle \tilde{e} \rangle_t \rangle_{xy} \rangle_{x_0 y_0} + \eta\omega\langle\langle\langle \tilde{e} \rangle_t \rangle_{xy} \rangle_{x_0 y_0} = \langle\langle \tilde{\pi}_{inj} \rangle_t \rangle_{x_0 y_0} \tag{35}$$

4. STOCHASTIC PLATE POWER FLOW EQUATIONS

The same scheme is proposed to derive the energy equations for the plate. The calculus will be given with few details, since the mathematical operations are similar to those explicitly valid for the membrane.

The governing equation for harmonic analysis of flexural motion of a homogeneous, isotropic, linear elastic plate, under the assumptions of small deflections, loaded by a time harmonic point force, is

$$\nabla^4 U - k^4(1 - \mathrm{i}\eta)U = F\delta(r_0). \tag{36}$$

$\eta \ll 1$ denotes the hysteretic damping coefficient, and k is the wave number. The solution of equation (36) is approximated to its far field plane wave components. The expression for the solution is

$$U = (A_x\,\mathrm{e}^{-\mathrm{i}k_x} + B_x\,\mathrm{e}^{\mathrm{i}k_x})(A_y\,\mathrm{e}^{-\mathrm{i}k_y} + B_y\,\mathrm{e}^{\mathrm{i}k_y}). \tag{37}$$

k_x and k_y are the complex wave number components which may be written as

$$k_x = k_{x1}(1 - \mathrm{i}\eta/4), \qquad k_y = k_{y1}(1 - \mathrm{i}\eta/4) \tag{38}$$

and

$$|k|^4 = \omega^2\sqrt{\rho h/D}. \tag{39}$$

The group velocity is defined as

$$c_g = 2\sqrt{\omega(D/\rho h)^{1/2}}, \tag{40}$$

where D is the flexural rigidity, h is the thickness and ρ is the density. To obtain a relationship between the expectations of time averaged energy density and intensity, the far field displacement given in equation (37) is introduced in the expressions for the energy variables. The co-ordinates x and y are replaced by the associated random co-ordinates

$\tilde{x} = x + \epsilon_x$ and $\tilde{y} = y + \epsilon_y$. The expressions for the energy variables are given in the Ph.D. Thesis of Bouthier [7]. The time averaged energy density is

$$\langle \tilde{e} \rangle_t = \frac{D}{4}\left(\frac{\partial^2 \tilde{U}}{\partial x^2}\left(\frac{\partial^2 \tilde{U}}{\partial x^2}\right)^* + \frac{\partial^2 \tilde{U}}{\partial y^2}\left(\frac{\partial^2 \tilde{U}}{\partial y^2}\right)^* + 2\nu \,\mathrm{Re}\left(\frac{\partial^2 \tilde{U}}{\partial x^2}\left(\frac{\partial^2 \tilde{U}}{\partial y^2}\right)^*\right) + 2(1-\nu)\frac{\partial^2 \tilde{U}}{\partial x\,\partial y}\left(\frac{\partial^2 \tilde{U}}{\partial x\,\partial y}\right)^* + \frac{\rho h}{D}\frac{\partial \tilde{U}}{\partial t}\left(\frac{\partial \tilde{U}}{\partial t}\right)^*\right). \tag{41}$$

The time averaged intensity integrated over the plate thickness may be expressed in terms of applied loads and motions. The components of the integrated intensity (W/m) are

$$\langle \tilde{I}_x \rangle_t = -\tilde{M}_{xx}\frac{\partial^2 \tilde{U}}{\partial x\,\partial t} - \tilde{M}_{xy}\frac{\partial^2 \tilde{U}}{\partial y\,\partial t} + \tilde{Q}_x\frac{\partial \tilde{U}}{\partial t}, \qquad \langle \tilde{I}_y \rangle_t = -\tilde{M}_{yy}\frac{\partial^2 \tilde{U}}{\partial y\,\partial t} - \tilde{M}_{yx}\frac{\partial^2 \tilde{U}}{\partial x\,\partial t} + \tilde{Q}_y\frac{\partial \tilde{U}}{\partial t}, \tag{42}$$

and

$$\tilde{M}_{xx} = -D\left(\frac{\partial^2 \tilde{U}}{\partial x^2} + \nu\frac{\partial^2 \tilde{U}}{\partial y^2}\right), \qquad \tilde{M}_{yy} = -D\left(\frac{\partial^2 \tilde{U}}{\partial y^2} + \nu\frac{\partial^2 \tilde{U}}{\partial x^2}\right),$$

$$\tilde{M}_{xy} = -D(1-\nu)\frac{\partial^2 \tilde{U}}{\partial x\,\partial y}, \qquad \tilde{M}_{yx} = \tilde{M}_{xy}, \qquad \tilde{Q}_x = \partial \tilde{M}_x/\partial x + \partial \tilde{M}_{xy}/\partial y,$$

$$\tilde{Q}_y = \partial \tilde{M}_y/\partial y - \partial \tilde{M}_{xy}/\partial x. \tag{43}$$

The expression for the expectation of the time averaged energy density may be written, in part, as

$$\begin{aligned}\langle\langle \tilde{e} \rangle_t \rangle_{xy} = {} & (D/4)(|k_x|^4 + |k_y|^4)\{|A_x|^2|A_y|^2\,\mathrm{e}^{-\eta(k_{x1}x + k_{y1}y)} + |A_x|^2|B_y|^2\,\mathrm{e}^{-\eta(k_{x1}x - k_{y1}y)} \\ & + |B_x|^2|A_y|^2\,\mathrm{e}^{\eta(k_{x1}x - k_{y1}y)} + |B_x|^2|B_y|^2\,\mathrm{e}^{\eta(k_{x1}x + k_{y1}y)}\} \\ & + A_xA_yB_x^*B_y^*|k_x|^4\,\mathrm{e}^{-\eta/2(k_{x1}x + k_{y1}y) - 2\mathrm{i}(k_{x1}x + k_{y1}y)}\,\mathrm{e}^{-2\sigma^2(k_{x1}^2 + k_{y1}^2)} + \cdots \\ \approx {} & (D/4)(|k_x|^4 + |k_y|^4)\{|A_x|^2|A_y|^2\,\mathrm{e}^{-\eta(k_{x1}x + k_{y1}y)} + |A_x|^2|B_y|^2\,\mathrm{e}^{-\eta(k_{x1}x - k_{y1}y)} \\ & + |B_x|^2|A_y|^2\,\mathrm{e}^{\eta(k_{x1}x - k_{y1}y)} + |B_x|^2|B_y|^2\,\mathrm{e}^{\eta(k_{x1}x + k_{y1}y)}\}.\end{aligned} \tag{44}$$

The high frequency approximations of the expectations of the temporal mean of the intensity components are

$$\begin{aligned}\langle\langle \tilde{I}_x \rangle_t \rangle_{xy} \approx {} & k_x(\omega/2)(k_x^2 + k_y^2 + |k_x|^2 + \nu|k_y|^2 + (1-\nu)|k_y|^2)\{|A_x|^2|A_y|^2\,\mathrm{e}^{-\eta(k_{x1}x + k_{y1}y))} \\ & + |A_x|^2|B_y|^2\,\mathrm{e}^{-\eta(k_{x1}x - k_{y1}y))} - |B_x|^2|A_y|^2\,\mathrm{e}^{\eta(k_{x1}x - k_{y1}y))} \\ & - |B_x|^2|B_y|^2\,\mathrm{e}^{\eta(k_{x1}x + k_{y1}y))}\},\end{aligned} \tag{45}$$

$$\begin{aligned}\langle\langle \tilde{I}_y \rangle_t \rangle_{xy} \approx {} & k_y(\omega/2)(k_x^2 + k_y^2 + |k_x|^2 + \nu|k_x|^2 + (1-\nu)|k_x|^2)\{|A_x|^2|A_y|^2\,\mathrm{e}^{-\eta(k_{x1}x + k_{y1}y))} \\ & - |A_x|^2|B_y|\,\mathrm{e}^{-\eta(k_{x1}x - k_{y1}y))} + |B_x|^2|A_y|^2\,\mathrm{e}^{\eta(k_{x1}x - k_{y1}y))} \\ & - |B_x|^2|B_y|^2\,\mathrm{e}^{\eta(k_{x1}x + k_{y1}y))}\}.\end{aligned} \tag{46}$$

Using equations (39, 40, 44–46), one can easily deduce the following high frequency relationship between $\langle\langle \vec{\tilde{I}} \rangle_t \rangle_{xy}$ and $\langle\langle \tilde{e} \rangle_t \rangle_{xy}$:

$$\langle\langle \vec{\tilde{I}} \rangle_t \rangle_{xy} = -(c_g^2/\eta\omega)\vec{\nabla}\langle\langle \tilde{e} \rangle_t \rangle_{xy} \tag{47}$$

According to Cremer and Heckl [4], one can write the dissipated power function of the energy density: $\langle\langle\tilde{\pi}_{diss}\rangle_t\rangle_{xy} = \eta\omega\langle\langle\tilde{e}\rangle_t\rangle_{xy}$. The governing energy differential equation may finally be written by using the energy balance, the expression of the intensity and the power dissipation relationships:

$$-(c_g^2/\eta\omega)\Delta\langle\langle\tilde{e}\rangle_t\rangle_{xy} + \eta\omega\langle\langle\tilde{e}\rangle_t\rangle_{xy} = \langle\pi_{inj}\rangle_t. \quad (48)$$

The expectation of the time averaged injected power may be evaluated in the same way as for the membrane. The time average of the input power is evaluated for a simple finite plate. The positions of the boundaries are random variables. The stochastic expectation is then expressed and it is shown that a high frequency approximation of this expression does not depend on the geometry of the plate. It is consequently assumed that the expression is valid for any type of geometry. Its value is finally given (without any calculus details) as

$$\langle\langle\tilde{\pi}_{inj}\rangle_t\rangle_{x_0y_0} \approx F^2/16\sqrt{D\rho h}. \quad (49)$$

This expression corresponds to the time averaged input power of the associated infinite plate. The final expression of the governing energy equation for the plate is

$$-(c_g^2/\eta\omega)\Delta\langle\langle\langle\tilde{e}\rangle_t\rangle_{xy}\rangle_{x_0y_0} + \eta\omega\langle\langle\langle\tilde{e}\rangle_{xy}\rangle_{x_0y_0} = \langle\langle\tilde{\pi}_{inj}\rangle_t\rangle_{x_0y_0}. \quad (50)$$

5. CONCLUSION

In this letter, an energy flow governing equation has been constructed for both plates and membranes. The unknown of the formulation is precisely defined as the expectation of the time averaged energy densities, and it is explicitly shown that the range of validity of the equation is restricted to the high frequency domain.

The interest of this formulation lies in the fact that the results of Bouthier and Bernhard are confirmed without using a "smoothing operation" whose signification was not clearly established, and the frequency range validity of their energy equation was not precisely given.

In other respects, the use of the "infinite input power" is justified with the same stochastic theory. The 'infinite input power' gives a simple relationship between the input force and the energy variable.

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